Quiz 9; Wednesday, October 25
MATH 110 with Professor Stankova
DSP
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Solutions

You have 10 minutes to complete the quiz. Calculators are not permitted. Please include all relevant calculations and explanations (unless stated otherwise).

1. (12 points) If v_1 and v_2 are eigenvectors of a linear operator T with eigenvalues λ_1 and λ_2 and $\lambda_1 \neq \lambda_2$, show that the T-cyclic subspace generated by $v_1 + v_2$ is 2-dimensional.

   Let W be the T-cyclic subspace generated by v_1+v_2. Note first that $v_1+v_2 \neq 0$ as they are necessarily linearly independent. Observe then that $T(v_1 + v_2) = \lambda_1 v_1 + \lambda_2 v_2 \in W$ and, as $\lambda_1 \neq \lambda_2$, it is not a multiple of $v_1 + v_2$. Thus, W contains at least two linearly independent vectors so $\dim(W) \geq 2$.

   On the other hand, note that $T^k(v_1 + v_2) = \lambda_1^k v_1 + \lambda_2^k v_2$ for any $k \geq 1$ so $W \subseteq \mathrm{span}(\{v_1, v_2\})$, showing that $\dim(W) \leq 2$. We conclude that $\dim(W) = 2$.

2. (1 + 1 + 1 points) Mark each statement as True or False. You do not need to show your work but a blank answer is worth 0 points and an incorrect answer is worth -1 point.

   (a) If T is a linear operator on an n-dimensional space, then $g(T) = 0$ for some polynomial $g(t)$ of degree n.

   True: Cayley-Hamilton.

   (b) If a linear operator T is diagonalizable, it must have distinct eigenvalues.

   False: Distinct eigenvalues are sufficient but not necessary for diagonalizability.

   (c) Let W_1 and W_2 be the T-cyclic subspaces generated by v_1 and v_2 respectively. If $v_1 \in W_2$, then $W_1 \subseteq W_2$.

   True: W_2 is also T-invariant so it contains all vectors of the form $T^k(v_1)$ and hence the entire W_1.